

ECCENTRICITY ON AN IMAGE CAUSED BY PROJECTION OF A CIRCLE AND A SPHERE

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Commission V, WG V/1

KEY WORDS: Eccentricity, Projection, Circle, Sphere, Ellipse, Formula

ABSTRACT:

Circular targets on a plane are often utilized in photogrammetry, particularly in close range photogrammetry, while spherical targets are sometimes utilized in industrial applications. Both a circle and a sphere are projected as an ellipse onto an image. There is an eccentricity on an image between the centre of the projected ellipse and the projected location of the centre of a circle or a sphere. Since only the centre of the projected ellipse is measured, the correction of the eccentricity is considered to be necessary for highly accurate measurement. This paper shows a process to derive general formulae to calculate an eccentricity of a circle and a sphere using the size and the location of a circle or a sphere, and the focal length, the position and the attitude of a camera. Furthermore the paper shows methods to estimate the eccentricity of a circle and a sphere from the equation of the projected ellipse of a circle or a sphere on an image.

1. INTRODUCTION

There are many factors affecting the accuracy of image measurement. Some of them should be considered when we execute image measurement. We have conducted a camera calibration in order to correct radial and decentering lens distortions since the era of analytical photogrammetry using analogue films. In recent years the correction of colour aberrations becomes indispensable for highly accurate measurement (Kaufmann and Ladstädter, 2005, Luhmann *et al.*, 2006a, Mallon and Whelan, 2007, Hattori *et al.*, 2011). In the era of digital photogrammetry the fact that sampling and quantization in creating a digital image affect the accuracy of image measurement using digital images was reported (Matsuoka *et al.*, 2011).

Furthermore there is an eccentricity of projection unless an object to be measured is a planar object lying on a plane parallel to the image plane of a camera. The eccentricity is a disparity on an image between the centre of the projected object and the projected location of the centre of an object. The correction of the eccentricity is considered to be necessary for highly accurate measurement.

In the meanwhile, circular targets on a plane are often utilized in photogrammetry, particularly in close range photogrammetry. Since a circle is radially symmetrical, circular targets are well suited for photogrammetric use such as camera calibration and 3D measurement. Although circular targets offer advantages of high image contrast, minimal size and high precision in image measurement, they have some drawbacks such as restricted visibility (typical: $\pm 45^\circ$) depending on light source and reflective properties (Luhmann, 2014).

On the other hand, in industrial and other technical applications spherical targets are used in order to maximize visibility and enable tactile probing by other systems, for example coordinate measurement machines (CMM) (Luhmann, 2014). Spherical

targets are used in the German guideline VDI/VDE 2634 Part 2 for acceptance testing and re-verification of optical 3D measuring systems (Verein Deutscher Ingenieure, 2002).

Both a circle on a plane and a sphere in space are projected as an ellipse onto an image. If a circle does not lie on a plane parallel to the image plane of a camera, or if the center of a sphere does not lie on the view vector of a camera, there is an eccentricity of projection to some extent. Figure 1 shows an eccentricity of projection of a circle.

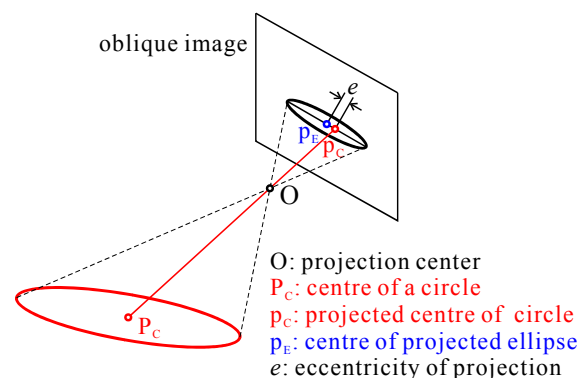


Figure 1. Eccentricity of projection of a circle

Regarding the eccentricity of a circle, Matsuoka *et al.* (2009) showed a formula calculating an eccentricity of a circle using the size and the location of a circle, and the focal length, the position and the attitude of a camera only when the circle lies on the horizontal plane. Moreover they presented merely the finally derived equation, although it would be convenient to show a process to derive the equation.

Luhmann (2014) reported the results of numerical simulations conducted in order to evaluate the effect of eccentricities to stereo and multi-image intersections. He quantified and investigated the eccentricities of circular targets and spherical

targets. Unfortunately he did not show a general formula calculating an eccentricity of a sphere.

This paper shows a process to derive general formulae to calculate an eccentricity of a circle and a sphere using the size and the location of a circle or a sphere, and the focal length, the position and the attitude of a camera. Furthermore the paper shows a method to estimate the eccentricity of a circle or a sphere from the equation of the projected ellipse of a circle or a sphere on an image.

2. PROJECTED LOCATION OF THE CENTRE OF A CIRCLE AND A SPHERE

The image coordinate system is assumed a planar rectangular coordinate system xy , while the object space coordinate system is assumed a spatial rectangular coordinate system XYZ . The camera coordinate system is a spatial rectangular coordinate system $\bar{X}\bar{Y}\bar{Z}$ as well.

Consider that an image of a circle C or a sphere S is taken by a camera with focal length f . Let $O(X_O, Y_O, Z_O)$ and $R(\omega, \phi, \kappa)$ be the position and the attitude of the camera in the object space coordinate system XYZ respectively. The position and the attitude of the camera in the camera coordinate system $\bar{X}\bar{Y}\bar{Z}$ are $O(0, 0, 0)$ and $R(0, 0, 0)$ respectively as well.

The rotation matrix $\mathbf{R}_{\omega\phi\kappa} = [a_{ij}]$ in the object space coordinate system XYZ can be represented by the following equation:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \omega & -\sin \omega \\ 0 & \sin \omega & \cos \omega \end{bmatrix} \begin{bmatrix} \cos \phi & 0 & \sin \phi \\ 0 & 1 & 0 \\ -\sin \phi & 0 & \cos \phi \end{bmatrix} \begin{bmatrix} \cos \kappa & -\sin \kappa & 0 \\ \sin \kappa & \cos \kappa & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (1)$$

Consider that the centre of a circle C is $P_C(X_C, Y_C, Z_C)$ in the object space coordinate system XYZ . The centre $P_C(\bar{X}_C, \bar{Y}_C, \bar{Z}_C)$ in the camera coordinate system $\bar{X}\bar{Y}\bar{Z}$ can be expressed by the following equation.

$$\begin{bmatrix} \bar{X}_C \\ \bar{Y}_C \\ \bar{Z}_C \end{bmatrix} = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix} \begin{bmatrix} X_C - X_O \\ Y_C - Y_O \\ Z_C - Z_O \end{bmatrix} \quad (2)$$

The projected location $p_c(x_c, y_c)$ in the image coordinate system xy of the centre $P_C(\bar{X}_C, \bar{Y}_C, \bar{Z}_C)$ in the camera coordinate system $\bar{X}\bar{Y}\bar{Z}$ can be expressed by the following form:

$$\begin{cases} x_c = -f \frac{\bar{X}_C}{\bar{Z}_C} \\ y_c = -f \frac{\bar{Y}_C}{\bar{Z}_C} \end{cases} \quad (3)$$

Similarly consider that the centre of a sphere S is $P_S(X_S, Y_S, Z_S)$ in the object space coordinate system XYZ . The centre $P_S(\bar{X}_S, \bar{Y}_S, \bar{Z}_S)$ in the camera coordinate system $\bar{X}\bar{Y}\bar{Z}$ can be expressed by the following equation.

$$\begin{bmatrix} \bar{X}_S \\ \bar{Y}_S \\ \bar{Z}_S \end{bmatrix} = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix} \begin{bmatrix} X_S - X_O \\ Y_S - Y_O \\ Z_S - Z_O \end{bmatrix} \quad (4)$$

The projected location $p_s(x_s, y_s)$ in the image coordinate system xy of the centre $P_S(\bar{X}_S, \bar{Y}_S, \bar{Z}_S)$ in the camera coordinate system $\bar{X}\bar{Y}\bar{Z}$ can be expressed by the following form:

$$\begin{cases} x_s = -f \frac{\bar{X}_S}{\bar{Z}_S} \\ y_s = -f \frac{\bar{Y}_S}{\bar{Z}_S} \end{cases} \quad (5)$$

3. GENERAL FORMULA TO CALCULATE AN ECCENTRICITY OF A CIRCLE

3.1 From the Size and the Location of a Circle

Let a circle C with radius R_c in the object space coordinate system XYZ be on the following plane H :

$$H: l(X - X_C) + m(Y - Y_C) + n(Z - Z_C) = 0 \quad (6)$$

where

$$l^2 + m^2 + n^2 = 1 \quad (7)$$

The normal vector \mathbf{v} of the plane H in the object space coordinate system XYZ can be represented as follows:

$$\mathbf{v} = [l \quad m \quad n]^T \quad (8)$$

Here let the plane H be expressed by the following equation in the camera coordinate system $\bar{X}\bar{Y}\bar{Z}$.

$$H: \bar{l}(\bar{X} - \bar{X}_C) + \bar{m}(\bar{Y} - \bar{Y}_C) + \bar{n}(\bar{Z} - \bar{Z}_C) = 0 \quad (9)$$

where

$$\bar{l}^2 + \bar{m}^2 + \bar{n}^2 = 1 \quad (10)$$

The normal vector $\bar{\mathbf{v}}$ of the plane H in the camera coordinate system $\bar{X}\bar{Y}\bar{Z}$ can be expressed as follows:

$$\bar{\mathbf{v}} = [\bar{l} \quad \bar{m} \quad \bar{n}]^T \quad (11)$$

The following equation can be obtained as well.

$$\begin{bmatrix} \bar{l} \\ \bar{m} \\ \bar{n} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix} \begin{bmatrix} l \\ m \\ n \end{bmatrix} \quad (12)$$

Consider that the following transformation T from $(\bar{X}, \bar{Y}, \bar{Z})$ to $(\bar{X}', \bar{Y}', \bar{Z}')$ makes the normal vector $\bar{\mathbf{v}}$ parallel to the \bar{Z}' axis:

$$\begin{bmatrix} \bar{X}' \\ \bar{Y}' \\ \bar{Z}' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \alpha & -\sin \alpha \\ 0 & \sin \alpha & \cos \alpha \end{bmatrix} \begin{bmatrix} \cos \beta & 0 & \sin \beta \\ 0 & 1 & 0 \\ -\sin \beta & 0 & \cos \beta \end{bmatrix} \begin{bmatrix} \bar{X} \\ \bar{Y} \\ \bar{Z} \end{bmatrix} \quad (13)$$

Because the transformation T makes the normal vector \bar{v} parallel to the \bar{Z} axis, the following equation is obtained:

$$\begin{bmatrix} \bar{l} & \bar{m} & \bar{n} \end{bmatrix}^t = [-\cos\alpha \sin\beta \quad \sin\alpha \quad \cos\alpha \cos\beta]^t \quad (14)$$

Finally α and β are obtained as follows:

$$\begin{cases} \alpha = \tan^{-1} \left\{ \frac{\bar{m}}{\sqrt{\bar{l}^2 + \bar{n}^2}} \right\} \\ \beta = -\tan^{-1} \left(\frac{\bar{l}}{\bar{n}} \right) \end{cases} \quad (15)$$

Let b_{ij} be as follows:

$$\begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix} \quad (16)$$

Then the transformation T and the inverse of the transformation T are expressed as follows:

$$\begin{bmatrix} \bar{X}' \\ \bar{Y}' \\ \bar{Z}' \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} \begin{bmatrix} \bar{X} \\ \bar{Y} \\ \bar{Z} \end{bmatrix} \quad (17)$$

$$\begin{bmatrix} \bar{X} \\ \bar{Y} \\ \bar{Z} \end{bmatrix} = \begin{bmatrix} b_{11} & b_{21} & b_{31} \\ b_{12} & b_{22} & b_{32} \\ b_{13} & b_{23} & b_{33} \end{bmatrix} \begin{bmatrix} \bar{X}' \\ \bar{Y}' \\ \bar{Z}' \end{bmatrix} \quad (18)$$

Consider that the point Q($\bar{X}_Q, \bar{Y}_Q, \bar{Z}_Q$) on the circumference of the circle C in the camera coordinate system $\bar{X}\bar{Y}\bar{Z}$ is projected to q(x_Q, y_Q) in the image coordinate system xy. Then the following equation is obtained:

$$\begin{bmatrix} x_Q & y_Q & -f \end{bmatrix}^t = k_Q \begin{bmatrix} \bar{X}_Q & \bar{Y}_Q & \bar{Z}_Q \end{bmatrix}^t \quad (19)$$

where

$$k_Q = -\frac{f}{\bar{Z}_Q} \quad (20)$$

Substituting Equation (18) for Equation (19) and rearranging gives the following equation:

$$\begin{bmatrix} \bar{X}'_Q \\ \bar{Y}'_Q \\ \bar{Z}'_Q \end{bmatrix} = \frac{1}{k_Q} \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} \begin{bmatrix} x_Q \\ y_Q \\ -f \end{bmatrix} \quad (21)$$

where

$$\frac{1}{k_Q} = \frac{\bar{Z}'_Q}{b_{31}x_Q + b_{32}y_Q - b_{33}f} \quad (22)$$

Substituting Equation (22) for Equation (21) gives the following equation:

$$\begin{bmatrix} \bar{X}'_Q \\ \bar{Y}'_Q \end{bmatrix} = \frac{\bar{Z}'_Q}{b_{31}x_Q + b_{32}y_Q - b_{33}f} \begin{bmatrix} b_{11}x_Q + b_{12}y_Q - b_{13}f \\ b_{21}x_Q + b_{22}y_Q - b_{23}f \end{bmatrix} \quad (23)$$

The circle C transformed by the transformation T is a circle with radius R_C and centre $P_C(\bar{X}'_C, \bar{Y}'_C, \bar{Z}'_C)$ in the coordinate system $\bar{X}'\bar{Y}'\bar{Z}'$, and is perpendicular to the \bar{Z}' axis. Because the point Q($\bar{X}'_Q, \bar{Y}'_Q, \bar{Z}'_Q$) in the coordinate system $\bar{X}'\bar{Y}'\bar{Z}'$ is on the circumference of the circle C, the following equation is obtained:

$$\begin{cases} (\bar{X}'_Q - \bar{X}'_C)^2 + (\bar{Y}'_Q - \bar{Y}'_C)^2 = R_C^2 \\ \bar{Z}'_Q = \bar{Z}'_C \end{cases} \quad (24)$$

where

$$\begin{bmatrix} \bar{X}'_C \\ \bar{Y}'_C \\ \bar{Z}'_C \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} \begin{bmatrix} \bar{X}_C \\ \bar{Y}_C \\ \bar{Z}_C \end{bmatrix} \quad (25)$$

Substituting Equation (23) for Equation (24) gives the following equation:

$$\begin{aligned} & \left(\bar{Z}'_C \frac{b_{11}x_Q + b_{12}y_Q - b_{13}f}{b_{31}x_Q + b_{32}y_Q - b_{33}f} - \bar{X}'_C \right)^2 \\ & + \left(\bar{Z}'_C \frac{b_{21}x_Q + b_{22}y_Q - b_{23}f}{b_{31}x_Q + b_{32}y_Q - b_{33}f} - \bar{Y}'_C \right)^2 = R_C^2 \end{aligned} \quad (26)$$

By rearrange Equation (26) as to (x_Q, y_Q, f) the following quadratic equation is obtained:

$$\begin{aligned} & p_{xx}x_Q^2 + p_{yy}y_Q^2 + p_{ff}f^2 \\ & + 2p_{xy}x_Qy_Q + 2p_{xf}x_Qf + 2p_{yf}y_Qf = 0 \end{aligned} \quad (27)$$

where p_{xx} , p_{yy} , p_{ff} , p_{xy} , p_{xf} and p_{yf} are as follows:

$$\begin{cases} p_{xx} = \bar{Z}'_C{}^2 - 2(b_{11}b_{31}\bar{X}'_C\bar{Z}'_C + b_{21}b_{31}\bar{Y}'_C\bar{Z}'_C) \\ \quad + (\bar{X}'_C{}^2 + \bar{Y}'_C{}^2 - \bar{Z}'_C{}^2 - R_C^2)b_{31}^2 \\ p_{yy} = \bar{Z}'_C{}^2 - 2(b_{12}b_{32}\bar{X}'_C\bar{Z}'_C + b_{22}b_{32}\bar{Y}'_C\bar{Z}'_C) \\ \quad + (\bar{X}'_C{}^2 + \bar{Y}'_C{}^2 - \bar{Z}'_C{}^2 - R_C^2)b_{32}^2 \\ p_{ff} = \bar{Z}'_C{}^2 - 2(b_{13}b_{33}\bar{X}'_C\bar{Z}'_C + b_{23}b_{33}\bar{Y}'_C\bar{Z}'_C) \\ \quad + (\bar{X}'_C{}^2 + \bar{Y}'_C{}^2 - \bar{Z}'_C{}^2 - R_C^2)b_{33}^2 \\ p_{xy} = -\{(b_{11}b_{32} + b_{31}b_{12})\bar{X}'_C\bar{Z}'_C + (b_{21}b_{32} + b_{31}b_{22})\bar{Y}'_C\bar{Z}'_C\} \\ \quad + (\bar{X}'_C{}^2 + \bar{Y}'_C{}^2 - \bar{Z}'_C{}^2 - R_C^2)b_{31}b_{32} \\ p_{xf} = +\{(b_{11}b_{33} + b_{31}b_{13})\bar{X}'_C\bar{Z}'_C + (b_{21}b_{33} + b_{31}b_{23})\bar{Y}'_C\bar{Z}'_C\} \\ \quad - (\bar{X}'_C{}^2 + \bar{Y}'_C{}^2 - \bar{Z}'_C{}^2 - R_C^2)b_{31}b_{33} \\ p_{yf} = +\{(b_{12}b_{33} + b_{32}b_{13})\bar{X}'_C\bar{Z}'_C + (b_{22}b_{33} + b_{32}b_{23})\bar{Y}'_C\bar{Z}'_C\} \\ \quad - (\bar{X}'_C{}^2 + \bar{Y}'_C{}^2 - \bar{Z}'_C{}^2 - R_C^2)b_{32}b_{33} \end{cases} \quad (28)$$

Here substituting Equations (25), (16) and (14) for Equation (28) and rearranging gives the following equation:

$$\begin{cases} p_{xx} = +(\bar{l}^2 + \bar{m}^2)\bar{y}_c^2 + (\bar{n}^2 + \bar{l}^2)\bar{z}_c^2 + 2\bar{m}\bar{n}\bar{y}_c\bar{z}_c - \bar{l}^2 R_c^2 \\ p_{yy} = +(\bar{l}^2 + \bar{m}^2)\bar{x}_c^2 + (\bar{m}^2 + \bar{n}^2)\bar{z}_c^2 + 2\bar{n}\bar{l}\bar{z}_c\bar{x}_c - \bar{m}^2 R_c^2 \\ p_{ff} = +(\bar{n}^2 + \bar{l}^2)\bar{x}_c^2 + (\bar{m}^2 + \bar{n}^2)\bar{y}_c^2 + 2\bar{l}\bar{m}\bar{x}_c\bar{y}_c - \bar{n}^2 R_c^2 \\ p_{xy} = +\bar{l}\bar{m}\bar{z}_c^2 - \bar{n}\bar{l}\bar{y}_c\bar{z}_c - \bar{m}\bar{n}\bar{z}_c\bar{x}_c - (\bar{l}^2 + \bar{m}^2)\bar{x}_c\bar{y}_c - \bar{l}\bar{m}R_c^2 \\ p_{yf} = -\bar{n}\bar{l}\bar{y}_c^2 + \bar{l}\bar{m}\bar{y}_c\bar{z}_c + (\bar{n}^2 + \bar{l}^2)\bar{z}_c\bar{x}_c + \bar{m}\bar{n}\bar{x}_c\bar{y}_c + \bar{n}\bar{l}R_c^2 \\ p_{yf} = -\bar{m}\bar{n}\bar{x}_c^2 + (\bar{m}^2 + \bar{n}^2)\bar{y}_c\bar{z}_c + \bar{l}\bar{m}\bar{z}_c\bar{x}_c + \bar{n}\bar{l}\bar{x}_c\bar{y}_c + \bar{m}\bar{n}R_c^2 \end{cases} \quad (29)$$

Equation (27) shows an ellipse as which the circle C with radius R_c and centre $P_c(X_c, Y_c, Z_c)$ is projected onto the image. The centre $p_E(x_E, y_E)$, the semi-major axis a , the semi-minor axis b and the rotation angle θ of the major axis of the projected ellipse on the image take the following forms:

$$\begin{cases} x_E = -f \frac{p_{yy}p_{yf} - p_{xy}p_{yf}}{p_{xx}p_{yy} - p_{xy}^2} \\ y_E = -f \frac{p_{xx}p_{yf} - p_{xy}p_{yf}}{p_{xx}p_{yy} - p_{xy}^2} \end{cases} \quad (30)$$

$$\begin{cases} a = \frac{\sqrt{2w}}{\sqrt{(p_{xx} + p_{yy}) + \sqrt{(p_{xx} - p_{yy})^2 + 4p_{xy}^2}}} \\ b = \frac{\sqrt{2w}}{\sqrt{(p_{xx} + p_{yy}) - \sqrt{(p_{xx} - p_{yy})^2 + 4p_{xy}^2}}} \end{cases} \quad (31)$$

$$\theta = \frac{1}{2} \tan^{-1} \frac{2p_{xy}}{p_{xx} - p_{yy}} \quad (32)$$

where w is as follows:

$$\begin{aligned} w = & p_{xx}x_E^2 + p_{yy}y_E^2 + p_{ff}f^2 \\ & + 2p_{xy}x_Ey_E + 2p_{yf}x_Ef + 2p_{yf}y_Ef \end{aligned} \quad (33)$$

Hence the eccentricity $(\Delta x, \Delta y)$ of the circle C with radius R_c and centre $P_c(X_c, Y_c, Z_c)$ on the plane H can be expressed as follows:

$$\begin{cases} \Delta x = x_E - x_c = (-f) \left\{ \frac{p_{yy}p_{yf} - p_{xy}p_{yf}}{p_{xx}p_{yy} - p_{xy}^2} - \frac{\bar{X}_c}{\bar{Z}_c} \right\} \\ \Delta y = y_E - y_c = (-f) \left\{ \frac{p_{xx}p_{yf} - p_{xy}p_{yf}}{p_{xx}p_{yy} - p_{xy}^2} - \frac{\bar{Y}_c}{\bar{Z}_c} \right\} \end{cases} \quad (34)$$

Finally Equations (34), (29), (12), (2), and (1) provide the eccentricity $(\Delta x, \Delta y)$ of the circle C with radius R_c and centre $P_c(X_c, Y_c, Z_c)$ on the plane H represented by Equation (6).

3.2 From an Ellipse Projected onto an Image

From now on we are going to derive an eccentricity of a circle from an ellipse projected onto an image. First consider that a projected image of a circle C on an image is an ellipse E represented by the following equation:

$$E : p_{xx}x_Q^2 + p_{yy}y_Q^2 + p_{ff}f^2 + 2p_{xy}x_Qy_Q + 2p_{yf}x_Qf + 2p_{yf}y_Qf = 0 \quad (35)$$

If we can derive $(\bar{X}_c, \bar{Y}_c, \bar{Z}_c)$ from $p_{xx}, p_{yy}, p_{ff}, p_{xy}, p_{yf}$ and p_{yf} by using Equation (29), we can calculate an eccentricity $(\Delta x, \Delta y)$ of the circle C from the projected ellipse E by using Equation (34). Since it would be unable to derive $(\bar{X}_c, \bar{Y}_c, \bar{Z}_c)$ from $p_{xx}, p_{yy}, p_{ff}, p_{xy}, p_{yf}$ and p_{yf} analytically, we will obtain $(\bar{X}_c, \bar{Y}_c, \bar{Z}_c)$ numerically by Newton-Raphson method.

Now the following equations from f_1 to f_7 as to $(\bar{X}_c, \bar{Y}_c, \bar{Z}_c)$, R_c and $(\bar{l}, \bar{m}, \bar{n})$ are to be solved by Newton-Raphson method:

$$\begin{cases} f_1 : (\bar{l}^2 + \bar{m}^2)\bar{y}_c^2 + (\bar{n}^2 + \bar{l}^2)\bar{z}_c^2 + 2\bar{m}\bar{n}\bar{y}_c\bar{z}_c - \bar{l}^2 R_c^2 - p_{xx} = 0 \\ f_2 : (\bar{m}^2 + \bar{n}^2)\bar{z}_c^2 + (\bar{l}^2 + \bar{m}^2)\bar{x}_c^2 + 2\bar{n}\bar{l}\bar{z}_c\bar{x}_c - \bar{m}^2 R_c^2 - p_{yy} = 0 \\ f_3 : (\bar{n}^2 + \bar{l}^2)\bar{x}_c^2 + (\bar{m}^2 + \bar{n}^2)\bar{y}_c^2 + 2\bar{l}\bar{m}\bar{x}_c\bar{y}_c - \bar{n}^2 R_c^2 - p_{ff} = 0 \\ f_4 : +\bar{l}\bar{m}\bar{z}_c^2 - \bar{n}\bar{l}\bar{y}_c\bar{z}_c - \bar{m}\bar{n}\bar{z}_c\bar{x}_c - (\bar{l}^2 + \bar{m}^2)\bar{x}_c\bar{y}_c - \bar{l}\bar{m}R_c^2 - p_{xy} = 0 \\ f_5 : -\bar{n}\bar{l}\bar{y}_c^2 + \bar{l}\bar{m}\bar{y}_c\bar{z}_c + (\bar{n}^2 + \bar{l}^2)\bar{z}_c\bar{x}_c + \bar{m}\bar{n}\bar{x}_c\bar{y}_c + \bar{n}\bar{l}R_c^2 - p_{yf} = 0 \\ f_6 : -\bar{m}\bar{n}\bar{x}_c^2 + (\bar{m}^2 + \bar{n}^2)\bar{y}_c\bar{z}_c + \bar{l}\bar{m}\bar{z}_c\bar{x}_c + \bar{n}\bar{l}\bar{x}_c\bar{y}_c + \bar{m}\bar{n}R_c^2 - p_{yf} = 0 \\ f_7 : \bar{l}^2 + \bar{m}^2 + \bar{n}^2 - 1 = 0 \end{cases} \quad (36)$$

Here, since $(\bar{X}_c, \bar{Y}_c, \bar{Z}_c)$ and R_c are linearly dependent, consider that the centre of the circle is on the plane $\bar{Z} = -f$, that is $\bar{Z}_c = -f$.

To secure the convergence of Newton-Raphson method we adopt the following initial guess of $(\bar{X}_{c0}, \bar{Y}_{c0}, \bar{Z}_{c0})$, R_{c0} and $(\bar{l}_0, \bar{m}_0, \bar{n}_0)$:

$$\begin{cases} \bar{X}_{c0} = x_E = -f \frac{p_{yy}p_{yf} - p_{xy}p_{yf}}{p_{xx}p_{yy} - p_{xy}^2} \\ \bar{Y}_{c0} = y_E = -f \frac{p_{xx}p_{yf} - p_{xy}p_{yf}}{p_{xx}p_{yy} - p_{xy}^2} \\ \bar{Z}_{c0} = -f \end{cases} \quad (37)$$

$$R_{c0} = \sqrt{ab} \quad (38)$$

$$\begin{cases} \bar{l}_0^2 = \frac{-p_{xx}(\bar{X}_{c0}^2 - R_{c0}^2) + p_{yy}\bar{Y}_{c0}^2 + p_{ff}\bar{Z}_{c0}^2 - 2p_{yf}\bar{Y}_{c0}\bar{Z}_{c0}}{R_{c0}^2(\bar{X}_{c0}^2 + \bar{Y}_{c0}^2 + \bar{Z}_{c0}^2 - R_{c0}^2)} \\ \bar{m}_0^2 = \frac{+p_{xx}\bar{X}_{c0}^2 - p_{yy}(\bar{Y}_{c0}^2 - R_{c0}^2) + p_{ff}\bar{Z}_{c0}^2 - 2p_{yf}\bar{Z}_{c0}\bar{X}_{c0}}{R_{c0}^2(\bar{X}_{c0}^2 + \bar{Y}_{c0}^2 + \bar{Z}_{c0}^2 - R_{c0}^2)} \\ \bar{n}_0^2 = \frac{+p_{xx}\bar{X}_{c0}^2 + p_{yy}\bar{Y}_{c0}^2 - p_{ff}(\bar{Z}_{c0}^2 - R_{c0}^2) + 2p_{xy}\bar{X}_{c0}\bar{Y}_{c0}}{R_{c0}^2(\bar{X}_{c0}^2 + \bar{Y}_{c0}^2 + \bar{Z}_{c0}^2 - R_{c0}^2)} \end{cases} \quad (39)$$

The solution of Equation (36) by Newton-Raphson method gives the normal vector \mathbf{v} of the plane H on which the circle C places in addition to the eccentricity $(\Delta x, \Delta y)$ of the circle C. If the radius R_c of the circle C is known, we can know the centre $P_c(X_c, Y_c, Z_c)$ of the circle C.

4. GENERAL FORMULA TO CALCULATE AN ECCENTRICITY OF A SPHERE

4.1 From the Size and the Location of a Sphere

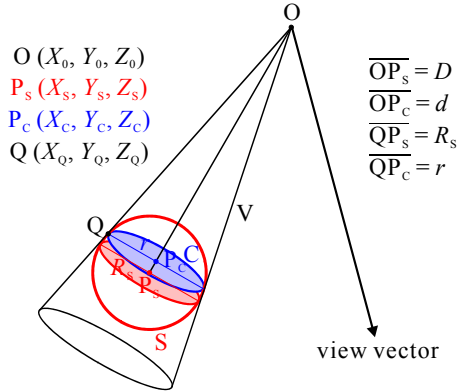


Figure 2. Sphere S and its small circle C

Consider that a right circular cone V with vertex $O(\bar{X}_0, \bar{Y}_0, \bar{Z}_0)$ in the camera coordinate system $\bar{X}\bar{Y}\bar{Z}$ is tangent to the sphere S. The tangent line to the sphere S is a small circle C of the sphere S. The circumference of the ellipse on the image is the projected circumference of the small circle C. The small circle C is perpendicular to the axis of the cone V. Let r and $P_c(\bar{X}_c, \bar{Y}_c, \bar{Z}_c)$ in the camera coordinate system $\bar{X}\bar{Y}\bar{Z}$ be the radius and the centre of the small circle C respectively. Consider that a point $Q(\bar{X}_q, \bar{Y}_q, \bar{Z}_q)$ in the camera coordinate system $\bar{X}\bar{Y}\bar{Z}$ is on the circumference of the small circle C.

The centre $P_c(\bar{X}_c, \bar{Y}_c, \bar{Z}_c)$ of the small circle C is represented as follows:

$$[\bar{X}_c \ \bar{Y}_c \ \bar{Z}_c]^t = \frac{d}{D} [\bar{X}_s \ \bar{Y}_s \ \bar{Z}_s]^t \quad (40)$$

where

$$\begin{cases} D = \sqrt{\bar{X}_s^2 + \bar{Y}_s^2 + \bar{Z}_s^2} \\ d = \sqrt{\bar{X}_c^2 + \bar{Y}_c^2 + \bar{Z}_c^2} \end{cases} \quad (41)$$

Because $\triangle OP_sQ$ and $\triangle QP_sP_c$ are similar, the following equation is obtained:

$$\frac{R_s}{D} = \frac{D-d}{R_s} \quad (42)$$

Rearranging Equation (42) gives the following equation:

$$d = D \left(1 - \frac{R_s^2}{D^2} \right) \quad (43)$$

Substituting Equation (43) for Equation (40) gives the following equation:

$$\begin{bmatrix} \bar{X}_c \\ \bar{Y}_c \\ \bar{Z}_c \end{bmatrix} = \left(1 - \frac{R_s^2}{D^2} \right) \begin{bmatrix} \bar{X}_s \\ \bar{Y}_s \\ \bar{Z}_s \end{bmatrix} \quad (44)$$

Because $\triangle OP_cQ$ and $\triangle QP_cP_s$ are similar, the following equation is obtained:

$$\frac{r}{d} = \frac{D-d}{r} \quad (45)$$

Rearranging Equation (45) gives the following equation:

$$r^2 = d(D-d) \quad (46)$$

Substituting Equation (42) for Equation (46) gives the following equation:

$$r^2 = R_s^2 \left(1 - \frac{R_s^2}{D^2} \right) \quad (47)$$

Consider that the following transformation T from $(\bar{X}, \bar{Y}, \bar{Z})$ to $(\bar{X}', \bar{Y}', \bar{Z}')$ makes the axis of the cone V parallel to the \bar{Z} axis:

$$\begin{bmatrix} \bar{X}' \\ \bar{Y}' \\ \bar{Z}' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \alpha & -\sin \alpha \\ 0 & \sin \alpha & \cos \alpha \end{bmatrix} \begin{bmatrix} \cos \beta & 0 & \sin \beta \\ 0 & 1 & 0 \\ -\sin \beta & 0 & \cos \beta \end{bmatrix} \begin{bmatrix} \bar{X} \\ \bar{Y} \\ \bar{Z} \end{bmatrix} \quad (48)$$

Let b_{ij} be as follows:

$$\begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \alpha & -\sin \alpha \\ 0 & \sin \alpha & \cos \alpha \end{bmatrix} \begin{bmatrix} \cos \beta & 0 & \sin \beta \\ 0 & 1 & 0 \\ -\sin \beta & 0 & \cos \beta \end{bmatrix} \quad (49)$$

Then the transformation T and the inverse of the transformation T are expressed as follows:

$$\begin{bmatrix} \bar{X}' \\ \bar{Y}' \\ \bar{Z}' \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} \begin{bmatrix} \bar{X} \\ \bar{Y} \\ \bar{Z} \end{bmatrix} \quad (50)$$

$$\begin{bmatrix} \bar{X} \\ \bar{Y} \\ \bar{Z} \end{bmatrix} = \begin{bmatrix} b_{11} & b_{21} & b_{31} \\ b_{12} & b_{22} & b_{32} \\ b_{13} & b_{23} & b_{33} \end{bmatrix} \begin{bmatrix} \bar{X}' \\ \bar{Y}' \\ \bar{Z}' \end{bmatrix} \quad (51)$$

Because the transformation T makes the axis of the cone V parallel to the \bar{Z} axis, the following equations are obtained:

$$\begin{bmatrix} \bar{X}_s \\ \bar{Y}_s \\ \bar{Z}_s \end{bmatrix} = \begin{bmatrix} -b_{31}D \\ -b_{32}D \\ -b_{33}D \end{bmatrix} = \begin{bmatrix} D \cos \alpha \sin \beta \\ -D \sin \alpha \\ -D \cos \alpha \cos \beta \end{bmatrix} \quad (52)$$

$$\begin{bmatrix} \bar{X}_c \\ \bar{Y}_c \\ \bar{Z}_c \end{bmatrix} = \begin{bmatrix} -b_{31}d \\ -b_{32}d \\ -b_{33}d \end{bmatrix} = \begin{bmatrix} d \cos \alpha \sin \beta \\ -d \sin \alpha \\ -d \cos \alpha \cos \beta \end{bmatrix} \quad (53)$$

Finally α and β are obtained as follows:

$$\begin{cases} \alpha = -\tan^{-1} \left\{ \frac{\bar{Y}_s}{\sqrt{\bar{X}_s^2 + \bar{Z}_s^2}} \right\} = -\tan^{-1} \left\{ \frac{\bar{Y}_c}{\sqrt{\bar{X}_c^2 + \bar{Z}_c^2}} \right\} \\ \beta = -\tan^{-1} \left(\frac{\bar{X}_s}{\bar{Z}_s} \right) = -\tan^{-1} \left(\frac{\bar{X}_c}{\bar{Z}_c} \right) \end{cases} \quad (54)$$

Consider that the point $Q(\bar{X}_Q, \bar{Y}_Q, \bar{Z}_Q)$ on the circumference of the small circle C is projected to $q(x_Q, y_Q)$ on the image. Then the following equation is obtained:

$$\begin{bmatrix} x_Q & y_Q & -f \end{bmatrix}^t = k_Q \begin{bmatrix} \bar{X}_Q & \bar{Y}_Q & \bar{Z}_Q \end{bmatrix}^t \quad (55)$$

where

$$k_Q = -\frac{f}{\bar{Z}_Q} \quad (56)$$

Substituting Equation (50) and Equation (51) for Equation (55) gives the following equation:

$$\begin{bmatrix} x_Q \\ y_Q \\ -f \end{bmatrix} = k_Q \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix} \begin{bmatrix} b_{11} & b_{21} & b_{31} \\ b_{12} & b_{22} & b_{32} \\ b_{13} & b_{23} & b_{33} \end{bmatrix} \begin{bmatrix} \bar{X}'_Q \\ \bar{Y}'_Q \\ \bar{Z}'_Q \end{bmatrix} \quad (57)$$

Here let c_{ij} be as follows:

$$\begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad (58)$$

Substituting Equation (58) for Equation (57) gives the following equation:

$$\begin{bmatrix} \bar{X}'_Q \\ \bar{Y}'_Q \\ \bar{Z}'_Q \end{bmatrix} = \frac{1}{k_Q} \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix} \begin{bmatrix} x_Q \\ y_Q \\ -f \end{bmatrix} \quad (59)$$

where

$$k_Q = -\frac{f}{c_{13}\bar{X}'_Q + c_{23}\bar{Y}'_Q + c_{33}\bar{Z}'_Q} = \frac{c_{31}x_Q + c_{32}y_Q - c_{33}f}{\bar{Z}'_Q} \quad (60)$$

Substituting Equation (60) for Equation (59) gives the following equation:

$$\begin{bmatrix} \bar{X}'_Q \\ \bar{Y}'_Q \end{bmatrix} = \frac{\bar{Z}'_Q}{c_{31}x_Q + c_{32}y_Q - c_{33}f} \begin{bmatrix} c_{11}x_Q + c_{12}y_Q - c_{13}f \\ c_{21}x_Q + c_{22}y_Q - c_{23}f \end{bmatrix} \quad (61)$$

The small circle C transformed by the transformation T is a circle with radius r and centre $P_C(0, 0, -d)$ in the coordinate system $\bar{X}'\bar{Y}'\bar{Z}'$, and is perpendicular to the \bar{Z}' axis. Accordingly the small circle C in the coordinate system $\bar{X}'\bar{Y}'\bar{Z}'$ is represented as follows:

$$\begin{cases} \bar{X}'^2 + \bar{Y}'^2 = r^2 \\ \bar{Z}' = -d \end{cases} \quad (62)$$

Because the point $Q(\bar{X}'_Q, \bar{Y}'_Q, \bar{Z}'_Q)$ is on the circumference of the small circle C, the following equation is obtained:

$$\begin{cases} \bar{X}'_Q{}^2 + \bar{Y}'_Q{}^2 = r^2 \\ \bar{Z}'_Q = -d \end{cases} \quad (63)$$

Substituting Equation (61) for Equation (63) gives the following equation:

$$\begin{aligned} & \left\{ (-d) \frac{c_{11}x_Q + c_{12}y_Q - c_{13}f}{c_{31}x_Q + c_{32}y_Q - c_{33}f} \right\}^2 \\ & + \left\{ (-d) \frac{c_{21}x_Q + c_{22}y_Q - c_{23}f}{c_{31}x_Q + c_{32}y_Q - c_{33}f} \right\}^2 = r^2 \end{aligned} \quad (64)$$

By rearrange Equation (64) as to (x_Q, y_Q, f) the following quadratic equation is obtained:

$$\begin{aligned} & p_{xx}x_Q^2 + p_{yy}y_Q^2 + p_{ff}f^2 \\ & + 2p_{xy}x_Qy_Q + 2p_{xf}x_Qf + 2p_{yf}y_Qf = 0 \end{aligned} \quad (65)$$

where p_{xx} , p_{yy} , p_{ff} , p_{xy} , p_{xf} and p_{yf} are as follows:

$$\begin{cases} p_{xx} = +d^2 - c_{31}^2(d^2 + r^2) \\ p_{yy} = +d^2 - c_{32}^2(d^2 + r^2) \\ p_{ff} = +d^2 - c_{33}^2(d^2 + r^2) \\ p_{xy} = -c_{31}c_{32}(d^2 + r^2) \\ p_{xf} = +c_{31}c_{33}(d^2 + r^2) \\ p_{yf} = +c_{32}c_{33}(d^2 + r^2) \end{cases} \quad (66)$$

Here substituting Equations (58), (53) and (41) for Equation (66) and rearranging gives the following equation:

$$\begin{cases} p_{xx} = + \frac{(\bar{Y}'_C + \bar{Z}'_C)(\bar{X}'_C + \bar{Y}'_C + \bar{Z}'_C) - r^2\bar{X}'_C{}^2}{(\bar{X}'_C + \bar{Y}'_C + \bar{Z}'_C)} \\ p_{yy} = + \frac{(\bar{Z}'_C + \bar{X}'_C)(\bar{X}'_C + \bar{Y}'_C + \bar{Z}'_C) - r^2\bar{Y}'_C{}^2}{(\bar{X}'_C + \bar{Y}'_C + \bar{Z}'_C)} \\ p_{ff} = + \frac{(\bar{X}'_C + \bar{Y}'_C)(\bar{X}'_C + \bar{Y}'_C + \bar{Z}'_C) - r^2\bar{Z}'_C{}^2}{(\bar{X}'_C + \bar{Y}'_C + \bar{Z}'_C)} \\ p_{xy} = - \frac{\bar{X}'_C\bar{Y}'_C(\bar{X}'_C + \bar{Y}'_C + \bar{Z}'_C) + r^2\bar{X}'_C\bar{Y}'_C}{(\bar{X}'_C + \bar{Y}'_C + \bar{Z}'_C)} \\ p_{xf} = + \frac{\bar{Z}'_C\bar{X}'_C(\bar{X}'_C + \bar{Y}'_C + \bar{Z}'_C) + r^2\bar{Z}'_C\bar{X}'_C}{(\bar{X}'_C + \bar{Y}'_C + \bar{Z}'_C)} \\ p_{yf} = + \frac{\bar{Y}'_C\bar{Z}'_C(\bar{X}'_C + \bar{Y}'_C + \bar{Z}'_C) + r^2\bar{Y}'_C\bar{Z}'_C}{(\bar{X}'_C + \bar{Y}'_C + \bar{Z}'_C)} \end{cases} \quad (67)$$

Equation (65) shows an ellipse as which the sphere S with radius R_s and centre $P_s(X_s, Y_s, Z_s)$ is projected onto the image. The centre $p_E(x_E, y_E)$, the semi-major axis a , the semi-minor axis b and the rotation angle θ of the major axis of the projected ellipse on the image take the following forms:

$$\begin{cases} x_E = -f \frac{p_{yy}p_{xf} - p_{xy}p_{yf}}{p_{xx}p_{yy} - p_{xy}^2} \\ y_E = -f \frac{p_{xx}p_{yf} - p_{xy}p_{xf}}{p_{xx}p_{yy} - p_{xy}^2} \end{cases} \quad (68)$$

$$\begin{cases} a = \sqrt{\frac{2w}{(p_{xx} + p_{yy}) + \sqrt{(p_{xx} - p_{yy})^2 + 4p_{xy}^2}}} \\ b = \sqrt{\frac{2w}{(p_{xx} + p_{yy}) - \sqrt{(p_{xx} - p_{yy})^2 + 4p_{xy}^2}}} \end{cases} \quad (69)$$

$$\theta = \frac{1}{2} \tan^{-1} \frac{2p_{xy}}{p_{xx} - p_{yy}} \quad (70)$$

where w is as follows:

$$\begin{aligned} w = & p_{xx}x_E^2 + p_{yy}y_E^2 + p_{ff}f^2 \\ & + 2p_{xy}x_Ey_E + 2p_{xf}x_Ef + 2p_{yf}y_Ef \end{aligned} \quad (71)$$

Hence the eccentricity $(\Delta x, \Delta y)$ of the sphere S with radius R_s and centre $P_s(X_s, Y_s, Z_s)$ can be expressed as follows:

$$\begin{cases} \Delta x = x_E - x_s = (-f) \left\{ \frac{p_{yy}p_{xf} - p_{xy}p_{yf}}{p_{xx}p_{yy} - p_{xy}^2} - \frac{\bar{X}_s}{\bar{Z}_s} \right\} \\ \Delta y = y_E - y_s = (-f) \left\{ \frac{p_{xx}p_{yf} - p_{xy}p_{xf}}{p_{xx}p_{yy} - p_{xy}^2} - \frac{\bar{Y}_s}{\bar{Z}_s} \right\} \end{cases} \quad (72)$$

Finally Equations (72), (67), (47), (41), (44), (4), and (1) provide the eccentricity $(\Delta x, \Delta y)$ of the sphere S with radius R_s and centre $P_s(X_s, Y_s, Z_s)$.

4.2 From an Ellipse Projected onto an Image

From now on we are going to derive an eccentricity of a sphere from an ellipse projected onto an image. First consider that a projected image of a sphere S onto an image is an ellipse E represented by the following equation:

$$E : p_{xx}x_Q^2 + p_{yy}y_Q^2 + p_{ff}f^2 + 2p_{xy}x_Qy_Q + 2p_{xf}x_Qf + 2p_{yf}y_Qf = 0 \quad (73)$$

If we can derive $(\bar{X}_C, \bar{Y}_C, \bar{Z}_C)$ from $p_{xx}, p_{yy}, p_{ff}, p_{xy}, p_{xf}$ and p_{yf} by using Equation (67), we can calculate the eccentricity $(\Delta x, \Delta y)$ of the sphere S from the projected ellipse E by using Equation (72). Fortunately $(\bar{X}_C, \bar{Y}_C, \bar{Z}_C)$ is able to be derived from $p_{xx}, p_{yy}, p_{ff}, p_{xy}, p_{xf}$ and p_{yf} analytically. Since $(\bar{X}_C, \bar{Y}_C, \bar{Z}_C)$ and r are linearly dependent, $(\bar{X}_C, \bar{Y}_C, \bar{Z}_C)$ and r are represented using a scale factor s as follows:

$$\begin{cases} \bar{Z}_C = -s \sqrt{\frac{1}{3} \left\{ \frac{(p_{xx} + p_{yy} + p_{ff})(p_{xf}p_{yf})^2}{(p_{xy}p_{xf})^2 + (p_{xy}p_{yf})^2 + (p_{xf}p_{yf})^2} - \frac{p_{xf}p_{yf}}{p_{xy}} \right\}} \\ \bar{X}_C = -s \frac{p_{xy}}{p_{yf}} \bar{Z}_C \\ \bar{Y}_C = -s \frac{p_{xy}}{p_{xf}} \bar{Z}_C \end{cases} \quad (74)$$

$$r = \sqrt{2d^2 - (p_{xx} + p_{yy} + p_{ff})} \quad (75)$$

where

$$d^2 = \bar{X}_C^2 + \bar{Y}_C^2 + \bar{Z}_C^2 \quad (41)$$

Now we can calculate an eccentricity $(\Delta x, \Delta y)$ of the sphere S from the projected ellipse E by substituting Equation (74) for Equation (72).

Moreover substitution Equation (75) for Equation (46) and rearranging gives the following equation:

$$D = \frac{3d^2 - (p_{xx} + p_{yy} + p_{ff})}{d} \quad (76)$$

The radius R_s of the sphere S can be obtained as follows:

$$R_s = \sqrt{\frac{\{2d^2 - (p_{xx} + p_{yy} + p_{ff})\} \{3d^2 - (p_{xx} + p_{yy} + p_{ff})\}}{d^2}} \quad (77)$$

Then the centre $P_s(\bar{X}_s, \bar{Y}_s, \bar{Z}_s)$ in the camera coordinate system $\bar{X}\bar{Y}\bar{Z}$ of the sphere S is obtained as follows:

$$\begin{bmatrix} \bar{X}_s \\ \bar{Y}_s \\ \bar{Z}_s \end{bmatrix} = \frac{D}{d} \begin{bmatrix} \bar{X}_C \\ \bar{Y}_C \\ \bar{Z}_C \end{bmatrix} \quad (78)$$

Equations (76), (77) and (78) contain the scale factor s . If the radius R_s of the sphere S is known, we can estimate the scale factor s and finally we can know the centre $P_s(X_s, Y_s, Z_s)$ of the sphere S .

5. TRIAL CALCULATIONS OF ECCENTRICITIES

We conducted trial calculations of eccentricities of circular targets and spherical targets mostly according to Luhmann's simulation (2014). Since some errata were found in his paper, we modified some of imaging conditions. Therefore obtained results in the study were not necessary the same as Luhmann's simulation results.

5.1 Imaging Scenario

The assumptions of an imaging scenario by a digital single-lens reflex (DSLR) camera are as follows:

- (1) Camera
 - a) Image size: 4288 x 2848 pixels
 - b) Unit pixel size: 5.5 μ m
 - c) $f = 16$ mm
 - d) $(X_0, Y_0, Z_0) = (0 \text{ mm}, 0 \text{ mm}, 0 \text{ mm})$
 - e) $(\omega, \phi, \kappa) = (0, 0, 0)$ and ϕ varies from -35° to $+35^\circ$
- (2) Targets
 - a) R_c and R_s vary from 1 mm to 10 mm.
 - b) A circular target and a spherical target are located at P4 (0 mm, 0 mm, -500 mm).
 - c) A circular target and a spherical target are located at P6 (360 mm, 240 mm, -600 mm).
 - d) The circular targets lie on a plane tilted relative to the image plane by angles $\alpha = 20^\circ$ and $\beta = 10^\circ$.

Figures 3 and 4 show the semi-major axes a and the semi-minor axes b of ellipses on the acquired images. Figure 3 shows a and b against view angles ϕ of the camera where the circular target with $R_c = 5$ mm and the spherical target with $R_s = 5$ mm are located at P4. Figure 4 shows a and b against radii R_c, R_s of the circular target and the spherical target located at P6 where $\phi = 0^\circ$.

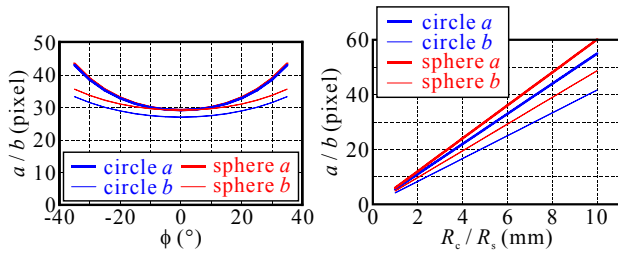


Figure 3. Semi-major axis a and semi-minor axis b vs. view angle ϕ where $R_c = R_s = 5\text{mm}$

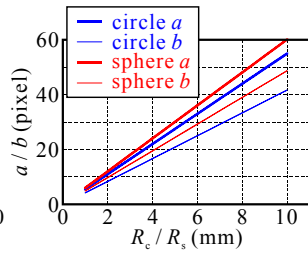


Figure 4. Semi-major axis a and semi-minor axis b vs. radius R_c, R_s where $\phi = 0^\circ$

5.2 Calculated Eccentricities

Figure 5 shows calculated eccentricities ($\Delta x, \Delta y$) against view angles ϕ of the camera where the circular target and the spherical target are located at P4 and $R_c = R_s = 5\text{mm}$. Figure 6 shows calculated eccentricities ($\Delta x, \Delta y$) against radii R_c, R_s of the circular target and the spherical target located at P6 where $\phi = 0^\circ$.

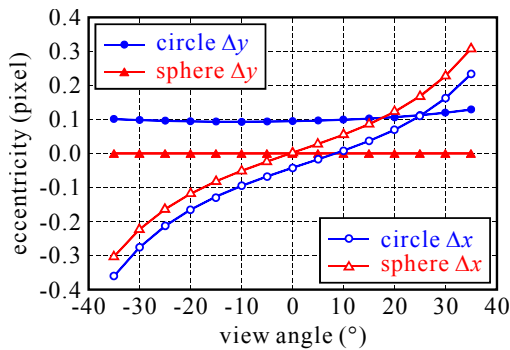


Figure 5. Eccentricity vs. view angle at P4 where $R_c = R_s = 5\text{mm}$

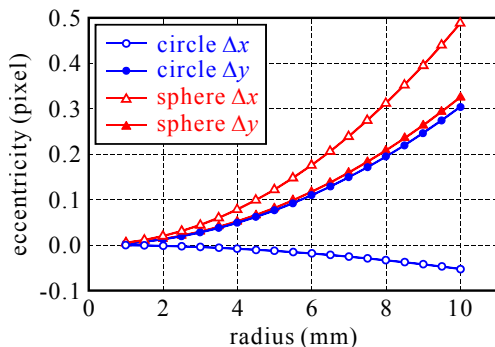


Figure 6. Eccentricity vs. radius at P6 where $\phi = 0^\circ$

5.3 Discussion on Eccentricities on Digital Camera Images

Figures 5 and 6 indicate that an eccentricity on an image acquired by the DSLR camera reaches almost a half pixel, which is considerably larger than the measurement accuracy of the centre of an ellipse on image (Luhmann *et al.*, 2006b).

Although the obtained formulae demonstrate that a smaller target produces a smaller eccentricity, a smaller target reduces the measurement accuracy of the centre of an ellipse on image as Luhmann *et al.* (2006b) mentioned. The magnitude of the effect of eccentricities on an image acquired by a digital camera depends on the resolution of the digital camera. As the

resolution of a digital camera becomes higher, the influence of eccentricities becomes larger. Finally the correction of eccentricities would become unavoidable.

6. CONCLUSIONS

The general formulae calculating an eccentricity of a circle by using the focal length, the position, the attitude of the camera, and the radius, the location of a circle on a plane was derived. The paper shows a numerical method to estimate the eccentricity of a circle from the equation of the projected ellipse of the circle on an image as well.

Moreover the general formulae calculating an eccentricity of a sphere by using the focal length, the position, the attitude of the camera, and the radius, the location of a sphere was derived. As for a sphere the paper shows general formulae to calculate the eccentricity of a sphere from the equation of the projected ellipse of the sphere on an image.

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